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Contrôle de Flux dans une fabrication en ligne

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RESUME

Dans ce papier, nous nous intéressons au contrôle de flux dans les lignes de production composées de séries de machines séparées par des stocks tampons. La capacité de chaque machine est constante et la demande est connue sur l'horizon complet. Les ruptures de stocks sont interdites. Le problème du contrôle de flux consiste à ajuster la production de chaque machine de manière à minimiser la somme des coûts engendrés par les stocks intermédiaires et le stock du produit fini. Nous établissons des propriétés de solutions optimales et proposons une solution analytique.

Flow Control of Production Lines *

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ABSTRACT

The paper addresses the flow control problem in production lines composed of a series of machines separated by buffers. We assume that the production capacity of each machine is constant and the the demand is known over the whole problem horizon. Backlogging is not allowed. The flow control problem consists in adjusting the production of the machines in order to minimize the total cost incurred by holding work in process and finished products. Properties of the optimal solutions are proposed. Based on these properties, we propose a simple analytical solution.

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+ This work was done while the first author was visiting INRIA

1. INTRODUCTION

A transfer line consists of a series of machines separated by buffers. Raw materials arrive from outside the system to the first machine. After being processed by the first machine, it queues in the first buffer waiting to be processed by the second machine. It continues in this manner through all machines and reaches the inventory of finished products after being processed by the last machine. The rate at which a machine produces is called production rate. We assume the the maximal production rate (or capacity) of each machine is constant.

Due to its importance in the control of manufacturing systems, flow control has been widely addressed for various types of production systems both in the deterministic case and the stochastic case (see [1-10]). In most work, mathematical programming models were proposed. Linear programming methods were used to find optimal flow control policies in deterministic case and dynamic programming approaches were used in the stochastic case.

This paper addresses the flow control in transfer lines. Only the deterministic case is considered. The objective is to establish some important characteristics of the optimal control policies. In particular, we establish the some conditions under which the intermediate buffers are always empty, i.e. zero work-in-process. Conditions under which a particular buffer is always empty are also established. Finally, a simple algorithm is proposed for computing the optimal control policy.

To the best of our knowledge, the results presented in this paper are new. As a matter of fact, analytical solutions to the single machine case were proposed in [3, 8]. However, no analytical solution has been proposed for the multi-stage case.

This paper is organized as follows. Section 2 describes the flow control model. Section 3 presents the results in the single machine case which are needed in solving the general case. Section 4 first addresses the demand feasibility, some special cases depending on the costs and the production capacity. It then proposes optimal control policies in the general case by using results of the special cases. Section 5 presents a numerical example and Section 6 is a conclusion.

2. PROBLEM SETTING

Let us consider a production line consisting of a series of n machines (M_1, M_2, ..., M_n) and n buffers (B_1, B_2, ..., B_n) where the buffer B_i is at the output of machine M_i. The buffer B_n contains finished products which can be used to meet the demand.

A discrete time model is considered in this paper. Let H be the number of elementary periods which is commonly called problem horizon.

The following notations will be used throughout the paper :

W_i : the maximal production capacity of machine M_i during each period
$u_{i,t}$: the production of M_i during period t
d_t : the demand during period t
$s_{i,t}$: the buffer level of B_i at the end of period t
c_i : the cost incurred by keeping one unit of product in B_i by the end of a period

We assume that the demand $(d_1, ..., d_H)$ is known over the whole horizon. The control variables to be determined are $u_{i,t}$. The vector $[s_{1,t}, s_{2,t}, \ldots, s_{n,t}]$ describes the state of the system by the end of period t.

We further assume that the buffers are initially empty, i.e.

$$s_{i,0} = 0, \forall i \tag{1}$$

The production capacity constraints can be expressed as follows :

$$0 \le u_{i,t} \le W_i, \forall i, \forall t \tag{2}$$

The buffer levels can be determined as follows :

$$s_{i,t} = s_{i,t-1} + u_{i,t} - u_{i+1,t}, \forall i < n, \forall t \tag{3}$$

and

$$s_{n,t} = s_{n,t-1} + u_{n,t} - d_t, \forall t \tag{4}$$

Since the levels of the intermediate buffers are positive and since backlogging is not allowed, we have :

$$s_{i,t} \ge 0, \quad \forall i, \forall t \tag{5}$$

The total cost incurred by the buffer levels is given by :

$$\sum_{i=1}^{n} \sum_{t=1}^{H} c_i s_{i,t}$$

The flow control problem consists in choosing $u_{i,t}$ $\forall i$ and $\forall t$ so as to

$$\text{minimize } \sum_{i=1}^{n} \sum_{t=1}^{H} c_i s_{i,t} \tag{6}$$

subject to the constraints (1) - (5).

3. SINGLE MACHINE CASE

We first notice that the single machine case has been solved (see [3, 8]). The purpose of this section is to provide a new solution to this case and to establish properties of this solution which will be needed later in solving the general case.

In the single machine case, we can neglect the index concerning the machines and the buffers and the problem becomes:

$$\text{minimize} \sum_{t=1}^{H} c\, s_t \tag{7}$$

subject to the following constraints :

$$0 \le u_t \le W,\ \forall t \tag{8}$$

$$s_t = s_{t-1} + u_t - d_t,\ \forall t \tag{9}$$

$$s_t \ge 0,\ \forall t \tag{10}$$

$$s_0 = 0 \tag{11}$$

As can be noticed, the optimal control policy is independent of the inventory holding cost c and we denote the problem (7) as **SMP(W, $[d_t]$)** in the following.

Let us consider a mapping $[\sigma_t] = \Phi(W,[d_t]) : \mathrm{IR} \times \mathrm{IR}^H \rightarrow \mathrm{IR}^{H+1}$ defined as follows:

$$\sigma_{t-1} = (\sigma_t + d_t - W)^+, \qquad \forall 1 \le t \le H \tag{12}$$

where

$$\sigma_H = 0$$

Consider also another mapping $[\upsilon_t] = \Psi(W,[d_t]) : \mathrm{IR} \times \mathrm{IR}^H \rightarrow \mathrm{IR}^H$ defined as follows:

$$\upsilon_t = \mathrm{Min}\{W, \sigma_t + d_t\}, \qquad \forall 1 \le t \le H \tag{13}$$

Theorem 1.

The demand is feasible iff $\sigma_0 = 0$. In case of feasible demand, $\Psi(W,[d_t])$ and $\Phi(W,[d_t])$ provide the optimal control policy and the optimal inventory trajectory respectively.

As it can be noticed, the optimal control policy consists in producing as late as possible and this policy leads to lowest inventory levels over the whole horizon. The proof of this theorem is based on the following two lemma.

Lemma 1.

$$\text{(a) } \sigma_{t-1} = \mathrm{Max}\left\{0, \mathop{\mathrm{Max}}_{t \le \tau \le H}\left\{\sum_{s=t}^{\tau} d_s - (\tau - t + 1)W\right\}\right\}, \ \forall 1 \le t \le H \tag{14}$$

(b) $\sigma_t = \sigma_{t-1} + \upsilon_t - d_t, \qquad \forall 1 \le t \le H$ (15)

(c) $0 \le \upsilon_t \le W, \qquad \forall 1 \le t \le H$ (16)

Proof of Lemma 1:

Claim (a) can be easily proved by induction and by using relation (13). Claim (c) is obvious from relations (12) and (13). Claim (b) can be proven by using relation (12) and the following relation :

$$\upsilon_t = \mathrm{Min}\{W, \sigma_t + d_t\} = \sigma_t + d_t - \mathrm{Max}\{0, \sigma_t + d_t - W\}$$

Q.E.D.

Lemma 2 :

In case of feasible demand, let $[s_t]$ be the inventory trajectory of a feasible solution to problem SMP(W, $[d_t]$). Then,

$$s_t \ge \sigma_t, \qquad \forall 0 \le t \le H$$

Proof of Lemma 2 :

Let us prove this lemma by contradiction. Assume that there exists an integer $t \ge 0$ such that

$$\sigma_t > s_t \ge 0$$

From lemma 1.a. there exists an integer t* with $t < t^* \le H$ such that :

$$\sigma_t = \sum_{s=t+1}^{\tau^*} d_s - (\tau^* - t)W > s_t$$

From constraints (8) and (9), we have :

$$s_{\tau^*} = s_t + \sum_{s=t+1}^{\tau^*} (u_s - d_s) \le s_t + \sum_{s=t+1}^{\tau^*} (u_s - d_s) \le s_t + \sum_{s=t+1}^{\tau^*} (W - d_s)$$

Combining the last two relations, we obtain :

$$s_{\tau^*} < \sum_{s=t+1}^{\tau^*} d_s - (\tau^* - t)W + \sum_{s=t+1}^{\tau^*} (W - d_s) = 0$$

which contradicts the feasibility of $[s_t]$.

Q.E.D.

Proof of Theorem 1 :

The optimality of $\Psi(W,[d_t])$ is a direct consequence of Lemma 2. Let us proof the feasibility. From Lemma 1, it is obvious that the mapping provides a feasible solution whenever s=0. Let us show that the demand is not feasible if $\sigma_0 > 0$. In this case, Lemma 1.a. implies that there exists a positive integer $\tau^* > 0$ such that

$$\sum_{s=1}^{\tau^*} d_s - \tau^* W > 0$$

As τ^*W is the maximal production capacity during the first τ^* periods and the buffer is initially empty, the above relation implies that the demand is greater than the maximal production capacity during the first τ^* periods and it cannot be satisfied.

Q.E.D.

From Lemma 1.a., it is obvious that the mapping is non-increasing in the machine capacity W.

Corollary 1.

The mapping $\Phi(W,[d_t])$ is non-increasing in W. More precisely, for any two positive numbers W_1 and W_2 with $W_1 \geq W_2 \geq 0$, it holds that:

$$\Phi(W_1,[d_t]) \leq \Phi(W_2,[d_t])$$

From lemma 1, it can be easily shown that if the demand is always lower than the capacity, then the inventory is always empty and the machine follows the demand.

Corollary 2.

If $d_t \leq W$ for all t, then $\Phi(W,[d_t]) = [0]$ and $\Psi(W,[d_t]) = [d_t]$.

4. GENERAL CASE

In this section, we first address the feasibility of the demand. We then address several special cases depending on the inventory holding costs or the production capacity. These results are then used to solve the general case.

4.1. Demand feasibility

The feasibility of the demand depends on the machine with the smallest production capacity called bottleneck machine. We prove in the following that the demand is feasible iff it is in the case of single machine whose capacity is the one of the bottleneck machine. The demand feasibility condition established in the single machine case can be used to check the demand feasibility of the general case.

Let

$$\underline{W} = \min_{1 \leq i \leq n} W_i$$

and

$$\underline{c} = \min_{1 \leq i \leq n} c_i$$

Theorem 2.

The demand is feasible iff the single machine problem SMP($\underline{W}$, $[d_t]$) has at least one feasible solution.

Proof :

Let M_m be the bottleneck machine, i.e.

$$W_m = \underset{1\le i\le n}{\text{Min}}\, W_i$$

For any feasible control $[u_{i,t}]$ of the general case, $[u_{m,t}]$ is clearly a feasible control of the single machine case SMP(W_m, $[d_t]$).

On the other hand, for any feasible control $[v_t]$ of SMP(W_m, $[d_t]$) and its related inventory trajectory $[\sigma_t]$, the control policy $[u_{i,t}]$ with $u_{i,t} = v_t$ $\forall i$ and $\forall t$ is a feasible control because of the following relations :

$$0 \le u_{i,t} = v_t \le W_m \le W_i,\ \forall i \text{ and } \forall t$$

$$s_{i,t} = 0,\ \ \forall 1\le i\le n\text{-}1,\ \forall t$$

$$s_{n,t} = \sigma_t \ge 0,\ \forall t$$

where $[s_{i,t}]$ are the inventory level trajectories related to the control $[u_{i,t}]$.

Q.E.D.

In the following, we assume that the demand is feasible.

4.2. Case : $W_n = \underline{W}$

It corresponds to the case in which the bottleneck machine is at the end of the production line.

Theorem 3.

If $W_n = \underline{W}$, then the optimal control policy is given by :

$$\left[u^*_{1,t}\right] = \left[u^*_{2,t}\right] = \ldots = \left[u^*_{n,t}\right] = \Psi\left(\underline{W},[d_t]\right)$$

The intermediate buffers are always empty under the optimal control policy, i.e.

$$s^*_{i,t} = 0, \qquad \forall 1 \le i \le n-1, \forall t$$

Proof :

The control $\left[u^*_{i,t}\right]$ is trivially a feasible control and the intermediate buffers are always empty, i.e.

$$s^*_{i,t} = 0, \qquad \forall 1 \le i \le n-1, \forall t$$

Let us prove the optimality. Consider a feasible control $[u_{i,t}]$ and its related inventory trajectories $[s_{i,t}]$. The control of machine M_n, i.e. $[u_{n,t}]$ is a feasible control of the single machine case SMP(W_n, $[d_t]$). From Theorem 1, we have :

$$s_{n,t} \ge s^*_{n,t}, \quad \forall t$$

Furthermore, since the inventory levels $s_{i,t}$ are nonnegative, then :

$$s_{i,t} \ge s^*_{i,t}, \qquad \forall 1 \le i \le n-1, \forall t$$

The above two relations imply the optimality of $\left[u_{i,t}^*\right]$.

Q.E.D.

4.3. Case : $c_n = \underline{c}$

It is the case where the output buffer of the production line has the lowest inventory holding cost.

Theorem 4.

If $c_n = \underline{c}$, then the optimal control policy is given by :

$$\left[u_{1,t}^*\right] = \left[u_{2,t}^*\right] = \ldots = \left[u_{n,t}^*\right] = \Psi\left(\underline{W}, [d_t]\right)$$

The intermediate buffers are always empty under the optimal control policy, i.e.

$$s_{i,t}^* = 0, \qquad \forall 1 \le i \le n-1, \forall t$$

Proof :

The control $\left[u_{i,t}^*\right]$ is trivially a feasible control and intermediate buffers are always empty, i.e.

$$s_{i,t}^* = 0, \qquad \forall 1 \le i \le n-1, \forall t$$

Let us prove the optimality. Let M_m be the bottleneck machine, i.e.

$$W_m = \underline{W}$$

Consider a feasible control $[u_{i,t}]$ and its related inventory trajectories $[s_{i,t}]$. The control of machine M_m, i.e. $[u_{m,t}]$ is a feasible control of the single machine case SMP(W_m, $[d_t]$). From Theorem 1, we have :

$$\sum_{i=m}^{n} s_{i,t} \ge \sum_{i=m}^{n} s_{i,t}^* = s_{n,t}^*, \qquad \forall t \tag{17}$$

Compare the total inventory holding costs of these two policies. We have :

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{t=1}^{H} c_i s_{i,t}^* &= \sum_{t=1}^{H} c_n s_{n,t}^* \\
&\le \sum_{t=1}^{H}\sum_{i=m}^{n} c_n s_{i,t} && \text{(by relation (17))} \\
&\le \sum_{t=1}^{H}\sum_{i=m}^{n} c_i s_{i,t} && \text{(since } c_n = \underline{c}\text{)} \\
&\le \sum_{i=1}^{n}\sum_{t=1}^{H} c_i s_{i,t}
\end{aligned}$$

Q.E.D.

4.4. Case of increasing inventory holding cost

In this case, we have

$$c_1 \le c_2 \le \ldots \le c_n$$

Theorem 5.

If the inventory holding cost c_i is increasing in i, then the optimal control policy is given by :

$$\left[u_{i,t}^*\right] = \Psi\left(\underset{i \le j \le n}{\mathrm{Min}} W_j, [d_t]\right), \quad \forall 1 \le i \le n$$

Proof :

In order to prove the feasibility and the optimality of $\left[u_{i,t}^*\right]$, let us consider a series of machines $\{M_{m(1)}, M_{m(2)}, \ldots, M_{m(J)}\}$ with increasing production capacity defined as follows :

$$W_{m(1)} = \underset{1 \le i \le n}{\mathrm{Min}} W_i$$

and

$$W_{m(j)} = \underset{m(j-1) < i \le n}{\mathrm{Min}} W_i, \qquad \forall 1 < j \le J$$

Of course, m(J) = n and by convention m(0) = 0. It obviously holds that :

$$\left[u_{i,t}^*\right] = \Psi\left(W_{m(j)}, [d_t]\right), \qquad \forall m(j-1) < i \le m(j) \tag{18}$$

and

$$s_{i,t}^* = 0, \qquad \forall m(j-1) < i < m(j), \forall t \tag{19}$$

(a) Feasibility of $\left[u_{i,t}^*\right]$

From the definition of $\left[u_{i,t}^*\right]$, it obviously satisfies the production capacity constraints of all the machines. Thanks to relation (19), we only need to prove that the inventory levels $s_{m(j),t}^*$ are non-negative.

Since m(J) = n, Theorem 1 implies that :

$$\left[s_{m(J),t}^*\right] = \Phi\left(W_{m(J)}, [d_t]\right)$$

which ensures that $s_{m(J),t}^* \ge 0$ for all t.

For any j < J, Theorem 1 implies that :

$$\left[\sum_{i=j}^{J} s_{m(i),t}^*\right] = \Phi\left(W_{m(j)}, [d_t]\right)$$

The inventory levels $s_{m(j),t}^*$ can thus be determined as follows:

$$\left[s_{m(j),t}^*\right] = \left[\sum_{i=j}^{J} s_{m(j),t}^*\right] - \left[\sum_{i=j+1}^{J} s_{m(j),t}^*\right] = \Phi\left(W_{m(j)}, [d_t]\right) - \Phi\left(W_{m(j+1)}, [d_t]\right)$$

As $W_{m(j)} \le W_{m(j+1)}$, Corollary 1 implies that :

$$\left[s_{m(j),t}^*\right] = \Phi\left(W_{m(j)}, [d_t]\right) - \Phi\left(W_{m(j+1)}, [d_t]\right) \ge 0$$

(b) Optimality of $\left[u^*_{i,t}\right]$

Let us consider another feasible control $[u_{i,t}]$ and its related inventory trajectories $[s_{i,t}]$. From Theorem 1, we have :

$$\sum_{i=m(j)}^{n} s_{i,t} \geq \sum_{i=j}^{J} s^*_{m(i),t}, \quad \forall 1 \leq j \leq J, \forall t$$

Since c_i is increasing in i, we have :

$$c_{m(1)} \sum_{i=m(1)}^{n} s_{i,t} \geq c_{m(1)} \sum_{i=1}^{J} s^*_{m(i),t}, \qquad \forall t$$

$$\left(c_{m(2)} - c_{m(1)}\right) \sum_{i=m(2)}^{n} s_{i,t} \geq \left(c_{m(2)} - c_{m(1)}\right) \sum_{i=2}^{J} s^*_{m(i),t}, \qquad \forall t$$

$$\ldots$$

$$\left(c_{m(J)} - c_{m(J-1)}\right) s_{m(J),t} \geq \left(c_{m(J)} - c_{m(J-1)}\right) s^*_{m(J),t}, \qquad \forall t$$

By summing up these relations, we obtain :

$$\sum_{j=1}^{J} c_{m(j)} \sum_{i=m(j)}^{m(j+1)-1} s_{i,t} \geq \sum_{j=1}^{J} c_{m(j)} s^*_{m(j),t}, \qquad \forall t \tag{20}$$

Let us compare the total inventory cost in each period. Then,

$$\begin{aligned}
\sum_{i=1}^{n} c_i s_{i,t} &\geq \sum_{i=m(1)}^{n} c_i s_{i,t} \quad (\text{since } s_{i,t} \geq 0) \\
&= \sum_{j=1}^{J} \sum_{i=m(j)}^{m(j+1)-1} c_i s_{i,t} \\
&\geq \sum_{j=1}^{J} c_{m(j)} \sum_{i=m(j)}^{m(j+1)-1} s_{i,t} \quad (\text{since } c_i \text{ is increasing}) \\
&\geq \sum_{j=1}^{J} c_{m(j)} s^*_{m(j),t} \quad (\text{from relation (20)}) \\
&= \sum_{i=1}^{n} c_i s^*_{i,t} \quad (\text{from relation (19)})
\end{aligned}$$

Q.E.D.

The following theorem shows that the optimal control policy can be expressed in another equivalent form. This new expression claims that in case of increasing inventory holding cost, each machine produce as late as possible in order to meet the demand from its immediate downstream machine and its production is independent of the control of its upstream machines.

Theorem 5'.

If the inventory holding cost c_i is increasing in i, then the optimal control policy is given by :

$$\left[u_{n,t}^{**}\right] = \Psi\left(W_n, \left[d_t\right]\right)$$

and

$$\left[u_{i,t}^{**}\right] = \Psi\left(W_i, \left[u_{i+1,t}^{**}\right]\right), \qquad \forall 1 \le i \le n-1$$

Proof :

We only consider the case of two machines, i.e. n = 2. The general case can be handled in a similar way.

We notice that the production of machine M_2 is the same as that under the control policy of Theorem 5, i.e.

$$\left[u_{2,t}^{*}\right] = \left[u_{2,t}^{**}\right] = \Psi\left(W_2, \left[d_t\right]\right)$$

Let us prove that $\left[u_{1,t}^{*}\right] = \left[u_{1,t}^{**}\right]$.We distinguish two cases : $W_1 \ge W_2$ or $W_1 < W_2$. Consider first the case of $W_1 \ge W_2$. In this case,

$$\left[u_{1,t}^{*}\right] = \Psi\left(\mathrm{Min}\{W_1, W_2\}, \left[d_t\right]\right) = \Psi\left(W_2, \left[d_t\right]\right)$$

$$\left[u_{1,t}^{**}\right] = \Psi\left(W_1, \left[u_{2,t}^{**}\right]\right)$$

As $u_{2,t}^{**} \le W_2 \le W_1$ for all t, Corollary 2 implies that :

$$\left[u_{1,t}^{**}\right] = \Psi\left(W_1, \left[u_{2,t}^{**}\right]\right) = \left[u_{2,t}^{**}\right] = \Psi\left(W_2, \left[d_t\right]\right) = \left[u_{1,t}^{*}\right]$$

Consider now the case of $W_1 < W_2$. In this case,

$$\left[u_{1,t}^{*}\right] = \Psi\left(\mathrm{Min}\{W_1, W_2\}, \left[d_t\right]\right) = \Psi\left(W_1, \left[d_t\right]\right)$$

$$\left[u_{1,t}^{**}\right] = \Psi\left(W_1, \left[u_{2,t}^{**}\right]\right)$$

We first remind that the single machine problem SMP(W_1, $\left[u_{2,t}^{**}\right]$) has at least one feasible solution. This can be easily proven by the following two facts : (i) $\left[u_{1,t}^{*}\right]$ is a feasible control of the problem SMP(W_1, $\left[u_{2,t}^{*}\right]$); and (ii) $\left[u_{2,t}^{*}\right] = \left[u_{2,t}^{**}\right]$. Theorem 1 implies that :

$$\left[s_{1,t}^{*}\right] \ge \left[s_{1,t}^{**}\right] \tag{21}$$

Remark also that $\left[u_{1,t}^{**}\right]$ is a feasible control of the single machine problem SMP(W_1, $[d_t]$). Since $\left[u_{1,t}^{*}\right] = \Psi\left(W_1, \left[d_t\right]\right)$, Theorem 1 implies that :

$$\left[s_{1,t}^{*} + s_{2,t}^{*}\right] \le \left[s_{1,t}^{**} + s_{2,t}^{**}\right]$$

Since $\left[s_{2,t}^{*}\right]=\left[s_{2,t}^{**}\right]=\Phi\left(W_2,[d_t]\right)$, we have :

$$\left[s_{1,t}^{*}\right]\le\left[s_{1,t}^{**}\right] \tag{22}$$

Relations (21) and (22) imply that :

$$\left[s_{1,t}^{*}\right]=\left[s_{1,t}^{**}\right]$$

which means that :

$$\left[u_{1,t}^{*}\right]=\left[u_{1,t}^{**}\right]$$

Q.E.D.

4.5. General case

This section derives optimal controls by applying results of the special cases previously addressed.

Theorem 6.

Let $\{B_{b(1)}, B_{b(2)}, \ldots, B_{b(J)}\}$ be a series of buffers with non-decreasing inventory holding cost defined as follows :

$$c_{b(1)}=\operatorname*{Min}_{1\le i\le n} c_i$$

$$c_{b(j)}=\operatorname*{Min}_{m(j-1)<i\le n} c_i, \quad \forall 1<j\le J$$

and

$$b(J)=n$$

The optimal control policy is given by :

$$\left[u_{n,t}^{*}\right]=\Psi\left(W_n,[d_t]\right)$$

$$\left[u_{b(i),t}^{*}\right]=\Psi\left(\underline{W}_i,\left[u_{b(i+1),t}^{*}\right]\right), \qquad \forall 1\le i\le J-1$$

$$\left[u_{i,t}^{*}\right]=\left[u_{b(j),t}^{*}\right], \qquad \forall b(j-1)<i<b(j)$$

where

$$\underline{W}_j=\operatorname*{Min}_{b(j-1)<i\le b(j)} W_i, \qquad \forall 1<j\le J$$

Remark that this theorem claims that the optimal control policy depends on the order of the inventory holding costs but it does not depend on their exact values.

Sketch of proof :

First, similar to the proof of theorem 4, it can be shown that under the optimal control policy, all the buffers except $\{B_{b(1)}, B_{b(2)}, \ldots, B_{b(J)}\}$ are always empty. This implies that the machines $\{M_{b(j)+1}, M_{b(j)+2}, \ldots, M_{b(j+1)}\}$ produces at the same speed.

As a result, the optimal control policy can be derived from an equivalent production line composed of J machines $(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_J)$ and J buffers $(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_J)$ where the inventory holding cost of buffer $\mathcal{B}_j$ is equal to $c_{b(j)}$ and the production capacity of machine $\mathcal{M}_j$ is equal to $\underline{W}_j$. The optimal control policy of this new line is given by Theorem 5. Finally, the production of the machines $\{M_{b(j)+1}, M_{b(j)+2}, \ldots, M_{b(j+1)}\}$ is equal to that of $\mathcal{M}_j$.

Q.E.D.

Similar to the proof of Theorem 6, the following equivalent form of the optimal solution can be derived.

Theorem 7.
Let $\{M_{m(1)}, M_{m(2)}, \ldots, M_{m(J)}\}$ be a series of machines with increasing capacity and $\{B_{b(1)}, B_{b(2)}, \ldots, B_{b(J)}\}$ a series of buffers with non-decreasing inventory holding cost defined as follows :

$$W_{m(j)} = \min_{b(j-1)<i\le n} W_i, \qquad \forall 1 \le j \le J$$

$$c_{b(j)} = \min_{m(j)\le i\le n} c_i, \qquad \forall 1 \le j \le J$$

and

$$b(0) = 0 \text{ and } b(J) = n$$

The optimal control policy is given by :

$$\left[u^*_{i,t}\right] = \Psi\left(W_{m(j)}, [d_t]\right), \qquad \forall b(j-1) < i \le b(j)$$

Q.E.D.

5. A NUMERICAL EXAMPLE

Consider an example of 12 machines and 10 elementary periods, i.e. n = 12 and H = 10. The production capacity and the inventory holding costs are given in Table 1.

i	1	2	3	4	5	6	7	8	9	10	11	12
W_i	7	8	5	10	11	8	10	12	9	11	10	11
c_i	6	10	7	8	3	6	7	5	9	9	10	7

Table 1: Production capacity and inventory holding cost

The demand is $[d_t]$ = [2, 1, 3, 3, 7, 2, 2, 10, 12, 4]. The minimal production capacity is equal to 5 and the demand is feasible.

We apply Theorem 7 to find the optimal solution. First, we obtain J = 3 and

$W_{m(1)} = 5$, $m(1) = 3$, $c_{b(1)} = 3$ and $b(1) = 5$

$W_{m(2)} = 8$, $m(2) = 6$, $c_{b(2)} = 5$ and $b(2) = 8$

$W_{m(3)} = 9$, $m(3) = 9$, $c_{b(3)} = 7$ and $b(3) = 12$

The optimal control policy is then obtained and given in Table 2.

t\i	1	2	3	4	5	6	7	8	9	10	11	12
0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0	0; 0
1	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2
2	0; 5	0; 5	0; 5	0; 5	4; 5	0; 1	0; 1	0; 1	0; 1	0; 1	0; 1	0; 1
3	0; 5	0; 5	0; 5	0; 5	6; 5	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3
4	0; 5	0; 5	0; 5	0; 5	8; 5	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3	0; 3
5	0; 5	0; 5	0; 5	0; 5	6; 5	0; 7	0; 7	0; 7	0; 7	0; 7	0; 7	0; 7
6	0; 5	0; 5	0; 5	0; 5	9; 5	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2	0; 2
7	0; 5	0; 5	0; 5	0; 5	6; 5	0; 8	0; 8	2; 8	0; 6	0; 6	0; 6	4; 6
8	0; 5	0; 5	0; 5	0; 5	3; 5	0; 8	0; 8	1; 8	0; 9	0; 9	0; 9	3; 9
9	0; 5	0; 5	0; 5	0; 5	0; 5	0; 8	0; 8	0; 8	0; 9	0; 9	0; 9	0; 9
10	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4	0; 4

Table 2: The optimal control policy ($s_{i,t}$, $u_{i,t}$)

6. CONCLUSION

This paper addresses the optimal flow control for transfer lines. We consider the case in which the production capacities are constant and the demand is known over the whole horizon. Properties of the optimal control policies have been established. In particular, some sufficient conditions under which the intermediate buffers are always empty have been proposed. We also established some sufficient conditions under which a particular buffer is always empty. Finally, a simple algorithm has been proposed for computing the optimal control policy.

Future research work consists in extending the results to other manufacturing systems. We believe that similar results can be obtained for assembly systems.

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